

0017-9310(94)00227-4

The heat flow rate in symmetrical two-dimensional conduction problems

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(Received 22 April 1994 and in final form 4 July 1994)

Abstract—Results of an analytical study on the heat flow rate in two-dimensional heat-conduction problems are presented. For conduction problems which satisfy some symmetry properties it is proven that the heat flow rate does not depend on the shape or dimensions of the geometry. For these problems the heat flow rate can be calculated without solving the distribution of the diffusing quantity. For the general problem it is shown that there is always a line which intersects isolines at a constant angle. Numerical results are presented to show the significance of the developed concepts.

INTRODUCTION

The diffusion of a physical quantity, such as temperature or concentration, is governed by the equation

$$\nabla D_\phi \nabla \phi = 0$$

in which ϕ is the transported quantity and D_ϕ the corresponding diffusion coefficient. Here, it is assumed that the problem is stationary and that no volume sources are present. If D_ϕ is independent of the position, the problem reduces to a Laplace equation for quantity ϕ .

Analytical solutions for this equation are available for a number of cases. Standard textbooks on heat conduction provide coverage of a large variety of analytical methods for solving heat-conduction problems [1, 2]. Generally, the presented analytical methods are only applicable or solvable for a confined set of problems and describe the problem in Cartesian, cylindrical or spherical coordinate systems. In addition, the methods aim to provide a full solution of the problem, viz. the distribution of quantity ϕ , whereas in general we are only interested in a specific aspect of the solution, e.g. the heat flow rate. Nowadays, most practical diffusion problems can be solved accurately by numerical methods and, consequently, the need for analytical solutions has decreased. Still, the analytical study of diffusion problems can increase insight into the behaviour and properties of the diffusion process. The obtained solutions may be useful for various practical problems.

This paper considers two-dimensional heat-conduction problems and, consequently, D_ϕ represents the thermal diffusivity $a = \lambda/\rho c_p$ and ϕ the temperature T . Here, we focus on the heat flow rate in a geometry bounded by two isothermal walls which are connected by adiabatic walls. Obviously, the obtained results are directly applicable to corresponding mass-

diffusion problems due to the correspondence between the equations for energy and mass diffusion. First, it is shown that the heat flow rate can be obtained analytically if the problem satisfies some symmetry properties. Afterwards, the general conduction problem is considered in order to show the existence of a special line in conduction problems. Finally, the insights obtained are discussed and enlightened by numerical results.

THE TWO-DIMENSIONAL HEAT-CONDUCTION PROBLEM

In the heat-conduction problems under consideration, the geometry is infinitely long in the third dimension. Its cross-section in the x - y -plane is bounded by two arbitrarily shaped isothermal walls which are connected by arbitrarily shaped adiabatic walls. By assuming that there are no gradients in the third dimension, the problem reduces to a two-dimensional heat-conduction problem. The thermal diffusivity of the material in the geometry (a) is assumed to be constant.

If the geometry and boundary conditions satisfy the following two properties, we call the problem *symmetrical*:

- (1) the geometry is line symmetric with respect to an axis of symmetry d ;
- (2) the boundary conditions for the temperature are skew-symmetric, i.e. isothermal walls are mirror images of adiabatic walls.

An example of a symmetrical heat-conduction problem is depicted in Fig. 1. Obviously, the set of diffusion problems satisfying these properties with respect to the symmetry is extensive. Although some of the problems can be solved in a straightforward way, it is not

NOMENCLATURE			
a	thermal diffusivity [$\text{m}^2 \text{s}^{-1}$]	β	angle of intersection between geometrical diagonals [$^\circ$]
c_p	specific heat at constant pressure [$\text{J kg}^{-1} \text{K}^{-1}$]	ΔT	temperature difference [K]
D_ϕ	diffusion coefficient for quantity ϕ	λ	thermal conductivity [$\text{W m}^{-1} \text{K}^{-1}$]
H	length of isothermal wall [m]	ξ, η	general coordinate
L	length of enclosure wall [m]	ρ	density [kg m^{-3}]
\mathbf{n}	unit normal vector	τ	unit tangential vector
Q	heat flow rate per unit length in third dimension [W m^{-1}]	φ	angle of inclination [$^\circ$]
T	temperature [K]	ϕ	general diffusing quantity
x, y	Cartesian coordinates [m].	Ψ	flux function [W m^{-1}].
Greek symbols		Subscripts	
α	angle of intersection between intersectional-diagonal and isotherms [$^\circ$]	c	at the cold wall
		h	at the hot wall.

always easy to determine analytically the solution for an arbitrarily shaped geometry.

In this paper the heat flow rate through the isothermal walls is calculated by using the symmetry of the problem. The actual solution of the temperature distribution is not considered. It will be proven that, for *all* symmetrical problems, the heat flow rate (per unit length in the third dimension) Q satisfies

$$Q = \lambda \Delta T \tag{1}$$

in which ΔT is the temperature difference between the isothermal walls and λ the thermal conductivity. Apparently, the heat flow rate does not depend on the shape or dimensions of the geometry.

In the proof we use a flux function Ψ for the heat flux in analogy with the stream function in hydrodynamics. For a line segment ds with unit normal vector \mathbf{n} , $d\Psi$ is given by

$$d\Psi = -\lambda \nabla T \cdot \mathbf{n} ds. \tag{2}$$

The relation between T and Ψ is analogous to the relation between the potential and stream function in potential theory:

$$\frac{\partial \Psi}{\partial x} = \lambda \frac{\partial T}{\partial y} \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = -\lambda \frac{\partial T}{\partial x}. \tag{3}$$

The property

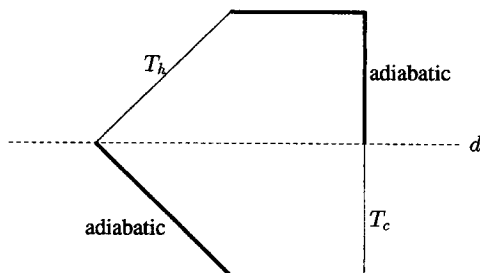


Fig. 1. Example of a symmetrical two-dimensional heat-conduction problem.

$$\nabla \Psi \cdot \nabla T = 0$$

shows that lines $\Psi = \text{constant}$ are perpendicular to isotherms.

PROOF OF $Q = \lambda \Delta T$ IN SYMMETRICAL PROBLEMS

In order to prove the relation $Q = \lambda \Delta T$ for symmetrical problems, we will focus on the heat-conduction problem in a rhombus. In Fig. 2, the geometry and boundary conditions under consideration are presented. The proof is divided into two steps. First, we will show that isotherms intersect the symmetry line d at an angle of 45° . By using this property, we can easily prove relation (1).

Theorem A

Isotherms for the heat-conduction problem presented in Fig. 2 intersect the symmetry line d at an angle of 45° .

Boundary conditions:

- $S_1: T = T_h$
- $S_2: (\partial T / \partial y) = 0$
- $S_3: T = T_c$
- $S_4: (\partial T / \partial y) = 0$

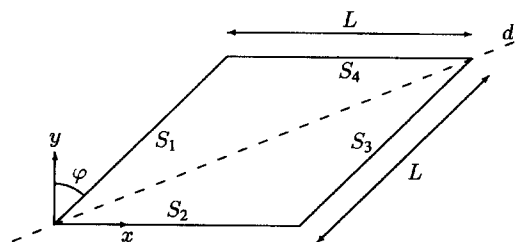


Fig. 2. Heat-conduction problem in a symmetrical geometry (rhombus).

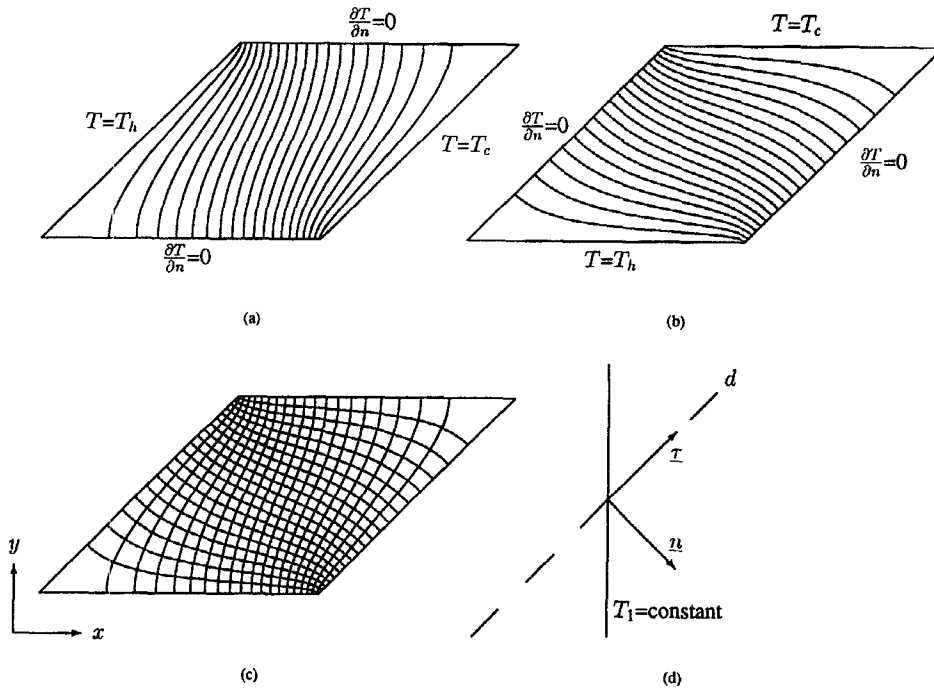


Fig. 3. Illustrations for the proof of $Q = \lambda \Delta T$ in a rhombus. (a) Original problem, isotherms T_1 . (b) Mirrored problem, isotherms T_2 . (c) Orthogonal isotherms (T_1, T_2). (d) Symmetry line d .

Proof

In Fig. 3(a) the solution T_1 of the conduction problem is given by drawing isotherms. Consider the heat-conduction problem obtained by mirroring the boundary conditions with respect to the symmetry line d [see Fig. 3(b)]. The equation describing the temperature distribution T_2 is given by

$\nabla^2 T_2 = 0$ with boundary conditions :

$$\begin{cases} S_1 : \frac{\partial T_2}{\partial n} = 0 & S_3 : \frac{\partial T_2}{\partial n} = 0 \\ S_2 : T_2 = T_h & S_4 : T_2 = T_c \end{cases}$$

The flux function of the original problem satisfies

$\nabla^2 \Psi_1 = 0$ with boundary conditions :

$$\begin{cases} S_1 : \frac{\partial \Psi_1}{\partial n} = 0 & S_3 : \frac{\partial \Psi_1}{\partial n} = 0 \\ S_2 : \Psi_1 = 0 & S_4 : \Psi_1 = Q \end{cases}$$

Consequently, the solutions for T_2 and Ψ_1 are related by

$$\frac{T_2 - T_c}{\Delta T} = - \frac{\Psi_1 - Q}{Q}$$

and the lines $T_2 = \text{constant}$ and $\Psi_1 = \text{constant}$ coincide. Since the lines $\Psi_1 = \text{constant}$ are orthogonal to the lines $T_1 = \text{constant}$, the isotherms for T_1 and T_2 are orthogonal [see Fig. 3(c)]. Since isotherms for T_2 can be obtained from mirroring the isotherms for T_1 , the fact that these lines are orthogonal shows us

that the isotherms have to intersect the symmetry line at an angle of 45° .

Theorem B

The heat flow rate through the hot wall is given by $Q = \lambda \Delta T$.

Proof

Since the isotherms T_1 intersect the symmetry line d at an angle of 45° , we have on d

$$\mathbf{n} \cdot \nabla T = \tau \cdot \nabla T$$

in which \mathbf{n} is the unit normal vector and τ the unit tangential vector on d [see Fig. 3(d)]. Integration over d gives

$$\begin{aligned} \int_d \mathbf{n} \cdot \nabla T \, ds &= \int_d \tau \cdot \nabla T \, ds \\ \Leftrightarrow \int_d - \frac{dT}{\lambda} &= \int_d \frac{dT}{\lambda} \, ds \\ \Leftrightarrow - \frac{Q}{\lambda} &= - \Delta T \end{aligned}$$

since by integrating over d we integrate both from $T = T_h$ to $T = T_c$ and from $\Psi = 0$ to $\Psi = Q$.

Obviously, the same proof is applicable to all potential and heat-conduction problems with the same symmetry in the geometry and boundary conditions. A few examples of symmetrical heat-conduction problems are given in Fig. 4.

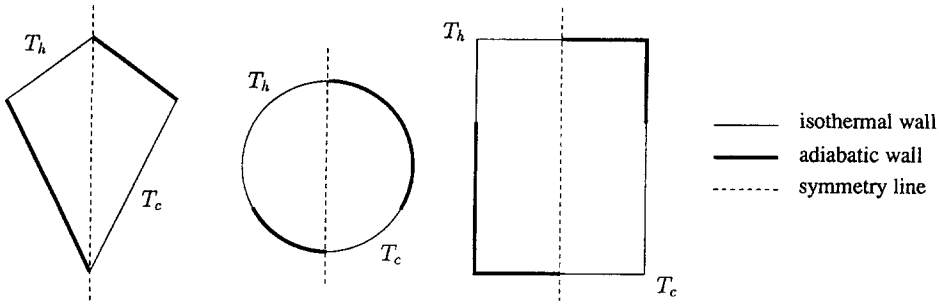


Fig. 4. Examples of heat-conduction problems for which $Q = \lambda \Delta T$.

The proof is not applicable to the general heat-conduction problem in two dimensions. However, as long as the geometry has one hot and one cold wall which are connected by adiabatic walls, there is always a line d which intersects the isotherms at a constant angle. This is shown in the next section. If the intersection angle is not known beforehand, we cannot find the heat flow rate from this line analytically.

A SPECIAL LINE IN TWO-DIMENSIONAL CONDUCTION PROBLEMS

In the previous section it was shown that for a set of heat-conduction problems, the heat flow rate can be calculated analytically and is independent of the dimensions of the geometry: $Q = \lambda \Delta T$. The fact that the symmetry line of the geometry intersects the isotherms at a constant angle of 45° is essential in the presented proof. If the line symmetry of the geometry is absent, the proof is not applicable. In this section, we consider the general two-dimensional heat-conduction problem for a geometry consisting of one hot and one cold wall connected by adiabatic walls. It will be shown that there is still a line which intersects the isotherms at a constant angle, the 'intersectional-diagonal'. If the intersection angle is not known beforehand, the heat flow rate cannot be calculated.

Consider the heat-conduction problem of Fig. 5. The geometry is not necessarily symmetrical. Again, we use the flux function Ψ , as defined in equation (2). The boundary conditions for the Laplace equations for T and Ψ ($\nabla^2 T = \nabla^2 \Psi = 0$) are given by

for T :

$$S_1 : T = T_h \quad S_2 : \frac{\partial T}{\partial n} = 0$$

$$S_3 : T = T_c \quad S_4 : \frac{\partial T}{\partial n} = 0$$

for Ψ :

$$S_1 : \frac{\partial \Psi}{\partial n} = 0 \quad S_2 : \Psi = 0$$

$$S_3 : \frac{\partial \Psi}{\partial n} = 0 \quad S_4 : \Psi = Q.$$

Now consider the transformation $(x, y) \rightarrow (\xi, \eta)$, with

$$\xi = \frac{\Psi}{\lambda \Delta T} \quad \text{and} \quad \eta = \frac{T - T_c}{\Delta T}$$

as depicted in Fig. 6. By relations (3), we can see that

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} \quad \text{and} \quad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x},$$

i.e. that the mapping is conformal (see e.g. refs. [3, 4]). The diagonal d of the rectangle in the (ξ, η) -plane intersects the lines $\eta = \text{constant}$ at a constant angle. Since the lines $\eta = \text{constant}$ correspond to isotherms in the original problem and since angles are conserved by conformal mappings, the line d in the (x, y) -plane intersects isotherms at a constant angle. The line d is not necessarily straight in the (x, y) -plane.

The line d connects the intersection points of the isothermal and adiabatic walls and might be called an 'intersectional-diagonal'. The angle of intersection between d and isotherms is given by

Boundary conditions:

$$S_1 : T = T_h$$

$$S_2 : (\partial T / \partial n) = 0$$

$$S_3 : T = T_c$$

$$S_4 : (\partial T / \partial n) = 0$$

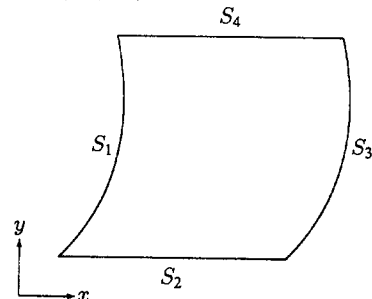


Fig. 5. Non-symmetrical two-dimensional heat-conduction problem.

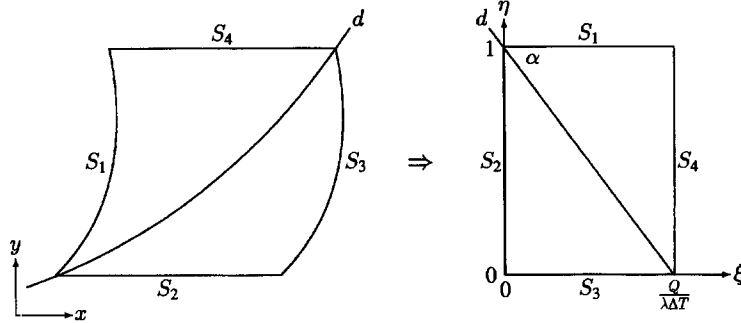


Fig. 6. Transformation from (x, y) - to (ξ, η) -coordinate system.

$$x = \tan^{-1} \frac{\lambda \Delta T}{Q}$$

and is not always known beforehand. If the geometry is line symmetric, the intersectional-diagonal coincides with the symmetry line and $\alpha = 45^\circ$. For these geometries we again have the previously derived relation $Q = \lambda \Delta T$.

DISCUSSION

In order to show the significance of the proofs presented above, we take a closer look at the heat-conduction problem for some geometries. For these problems the temperature distribution is solved numerically. By using a coordinate-invariant formulation of the diffusion problem [5], the Laplace equation for T is solved in a boundary-fitted coordinate system [6]. The equations are discretized by using a finite volume method as described by Patankar [7].

For the rhombic geometry of Fig. 2, calculations have been performed in which the inclination angle of the isothermal walls φ is changed from 0° to 60° . The resulting temperature distributions are presented in

Fig. 7 by plotting isotherms for the temperature levels $(T - T_c)/\Delta T = 0(0.05)1$. The solution for $\varphi = 0^\circ$ is straightforward; the linear temperature distribution gives rise to a heat flow rate $Q = \lambda(\Delta T/L)L = \lambda \Delta T$. For $\varphi \neq 0^\circ$ the isotherms are curved, since they are orthogonal to the adiabatic walls. Consequently, the local heat-flux distribution at the isothermal walls is not uniform. In corners with acute angles between the isothermal and adiabatic walls the flux is relatively low, whereas in corners with obtuse angles the flux is relatively high. Due to the symmetry in the problem, however, the heat flow rate is given by $Q = \lambda \Delta T$ for all φ . For $\varphi = 45^\circ$, this phenomenon has already been observed by Demirdžić *et al.* [8] from their numerical calculations.

If the dimension of the isothermal walls is larger than the dimension of the adiabatic walls, we obtain a parallelogram and the line symmetry disappears. Calculations have been performed for a parallelogrammic geometry in which the dimension of the isothermal walls (H) is different from the dimension of the adiabatic walls (L). For $\varphi = 0^\circ$, we again have the linear decrease of T between the hot and cold wall, giving $Q = (H/L)\lambda \Delta T$. In Fig. 8 the heat flow rate through the isothermal walls is given as a function of

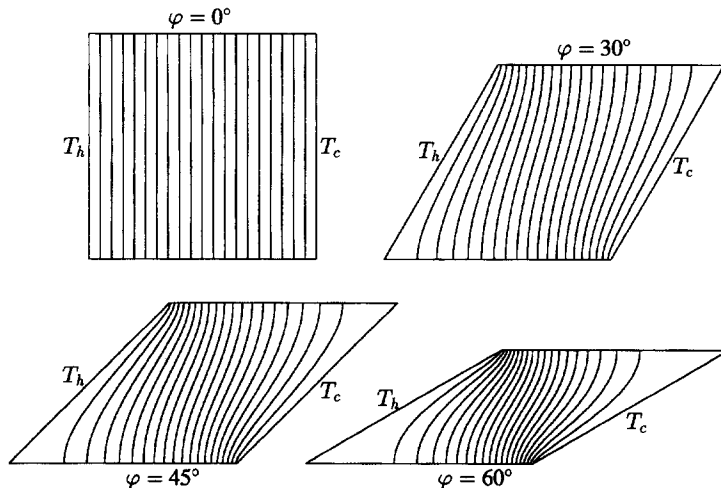


Fig. 7. Solutions of the heat-conduction problem in a rhombic geometry.

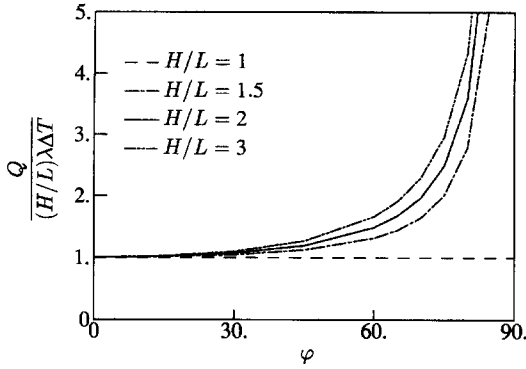


Fig. 8. Heat flow rate in parallelograms with aspect ratios $H/L = 1, 1.5, 2$ and 3 as a function of the inclination angle of the isothermal walls.

the inclination angle of the isothermal walls φ for $H/L = 1, 1.5, 2$ and 3 . Whereas $Q \approx (H/L)\lambda\Delta T$ for $\varphi < 45^\circ$, the heat flow rate increases rapidly if φ approaches 90° and $H > L$. Indeed, the situation $\varphi \uparrow 90^\circ$ and $H > L$ gives rise to a 'short-circuit' for the heat transfer and we have $Q \rightarrow \infty$. If the isothermal walls had been shorter than the adiabatic walls, the isothermal walls would not have touched if $\varphi \uparrow 90^\circ$

and we would have $Q \downarrow 0$. Only for the rhombic geometry δ° do the isothermal walls touch at a single point for $\varphi = 90^\circ$, giving a finite, positive heat flow rate.

For the parallelograms under consideration there still is a line d which intersects the isotherms at a constant angle, as proven previously. By calculating the temperature distribution for the original problem and for the problem in which the boundary conditions for T are reversed (see Fig. 3), two mutually orthogonal sets of isotherms appear. In fact, orthogonal grids for numerical calculations can be obtained this way [9]. We now have isotherms T_1 and T_2 in which lines $T_2 = \text{constant}$ correspond to lines $\Psi_1 = \text{constant}$. By connecting the points of intersection of identical isotherms T_1 and T_2 , the line which intersects the isotherms at a constant angle appears. In Fig. 9 this line d is drawn for a parallelogram with $H/L = 2$ and $\varphi = 45^\circ$. The line d is not straight. The intersection angle is not known beforehand, but can be obtained from the calculated value of Q . In the case of $H/L = 2$ and $\varphi = 45^\circ$ we find $\alpha \approx 23^\circ$. A second line d' can be obtained in the same way and is added in the plot.

Figure 10 shows that the angle of intersection between the intersectional-diagonals d and d' (2α)

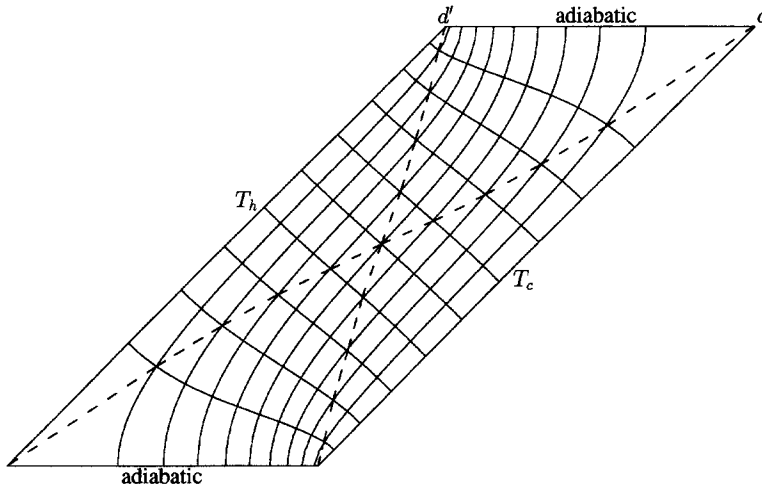


Fig. 9. Two lines in a parallelogram which intersect isotherms at a constant angle.

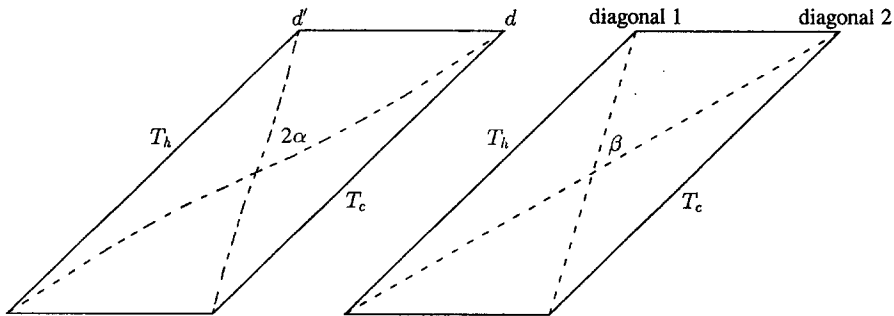


Fig. 10. Correspondence between intersection angle of intersectional-diagonals (2α) and intersection angle of geometrical diagonals (β).

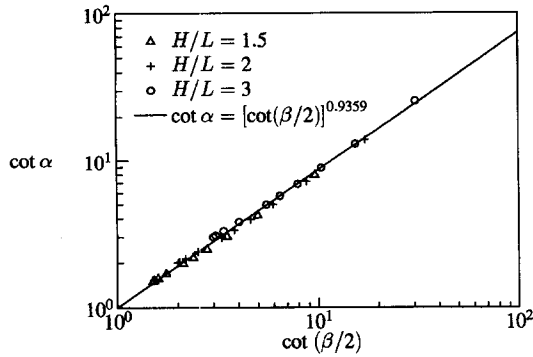


Fig. 11. Intersection angle between intersectional-diagonal and isotherms (α) as a function of half the intersection angle between geometrical diagonals ($\beta/2$) for parallelograms with $H/L = 1.5, 2$ and 3 .

approximately equals the angle of intersection between the geometrical diagonals of the parallelogram (β). Due to the proven relation $Q/\lambda\Delta T = \cot \alpha$, an estimation for the heat flow rate in a parallelogram is given by

$$Q \approx \lambda\Delta T \cot(\beta/2).$$

In Fig. 11, the calculated intersection angle between the intersectional-diagonal and isotherms (α) is plotted as a function of half the intersection angle between the geometrical diagonals ($\beta/2$). The plot shows that an even better estimation for the heat flow rate is given by

$$Q \approx \lambda\Delta T [\cot(\beta/2)]^{0.9359}$$

which gives rise to a maximum error of 8% for $1 \leq H/L \leq 3$ and $|\varphi| \leq 85^\circ$.

CONCLUSION

The two-dimensional potential problem has been considered by studying the heat-conduction problem

in a differentially heated geometry. The geometry is bounded by two isothermal walls which are connected by adiabatic walls. For the heat-conduction problem in a symmetrical geometry with 'skew-symmetric' boundary conditions for the energy equation, the heat flow rate has been proven to satisfy $Q = \lambda\Delta T$, i.e. the heat flow rate does not depend on the shape or dimensions of the geometry. For non-symmetric problems it has been shown that there is always a line through the geometry (an 'intersectional-diagonal') which intersects isotherms and flux lines at a constant angle. The obtained results have been enlightened by numerical results for the diffusion problem in rhombic and parallelogrammic geometries.

Acknowledgements—These investigations are supported by the Netherlands Foundation for Chemical Research (SON) with financial aid from the Netherlands Technology Foundation (STW).

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